Geometry of the BFV Theorem

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Abstract

We describe gauge-fixing at the level of virtual paths in the path integral as a non-symplectic BRST-type of flow on the path phase space. As a consequence a gauge-fixed, non-local symplectic structure arises. Restoring of locality is discussed. A pertinent anti-Lie-bracket and an infinite dimensional group of gauge fermions are introduced. Generalizations to Sp(2)-symmetric BLT-theories are made.

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I Introduction

Similarly to the Lagrangian BRST symmetry[1] for Yang-Mills theory, Batalin, Fradkin and Vilkovisky [2] developed a BRST formulation of an arbitrary Hamiltonian gauge theory with reducible first class constraints and possible open gauge algebra, nowadays known as BFV-BRST quantization[3]. The crucial ingredients turn out to be two Grassmann-odd objects, the nilpotent BRST charge Ω and the gauge fermion ψ. The BFV Theorem, which is the subject of this article, simply states that the partition function $\mathcal{Z}$ is independent of the choice of ψ.

The present work was originally motivated by a concern about the stability of time locality. It is a well-known fact that time has a special status in Hamiltonian theories. One assumes that all virtual paths in the path integral are parametrized by the one and same global time parameter. As a consequence, it makes sense to speak of equal-time Poisson brackets. In Hamiltonian theories, there is no interaction between two different time-slices. In other words, the symplectic two-form $\omega(t) = \omega(z(t), t)$ is a *function* of the coordinates taken in the very same time. This is often referred to as saying that the phase space Poisson structure is ultra-local in time. So the symplectic structure does not depend on the past nor the future. Nevertheless, in the standard proof[2] for the BFV Theorem one performs an infinitesimal flow generated by a bosonic BRST-type vector field

$$X(t) \ = \ \mu X_{\Omega(t)} \ , \tag{I.1}$$

where $\mu \sim \int_0^T dt \ \delta_{\rm ext}\psi(t)$ is a *functional.* The flow therefore carries information about the past and the future of the path. In a consistent geometric formulation such flows will violate the ultra-local ansatz for the two-form $\omega(t) = \omega(z(t), t)$. The BRST variation (I.1) is transporting well-defined ultra-local theories into non-local theories. Naively, one would expect that the pertinent BRST transformation would map ultra-local theories in ultra-local theories. However, this is not the case. Our analysis below shows that the non-locality arising in the canonical measure factor, the Pfaffian, remarkably can be recast into an ultra-local form, which turns out to be the gauge-fixing term in the action. This solves our original posed question, and it shows that the formalism is consistent.

Besides the time-locally problem mentioned above, we introduce and give a geometric description of the gauge fixing flow directly at the level of virtual paths in the path phase space.

The article is organized as follows: A pertinent anti-Lie-bracket, that turns the set of gauge fermions into an infinite-dimensional algebra is introduced in Section III. This algebra is interesting in its own right from a pure mathematical point of view. There we also discuss the realizations of the algebra of gauge fermions and its corresponding group. This paves the way for introduction of a gauge-fixing flow discussed in Section IV. With this in hand we can prove the BFV theorem in a path space formalism, see Section V. We have for completeness included a Section V.3 about the infinitesimal gauge-fixing flow. Finally, we shall in Section VI generalize the construction to $Sp(2)$-symmetric Hamiltonian theories[4]. Some of the

result in this article has been reported on the NATO Advanced Research Workshop, 1997, Zakopane, "New Developments in Quantum Field Theory".

II Lift to Path Space

The basic philosophy is to consider the collection $\mathcal{P}\Gamma$ of all paths as an infinite-dimensional manifold, *i.e.* a geometric object that does not depend on the choice of coordinates. For instance, given a volume form, the partition function is an integral over this manifold weighted with the Boltzmann factor. Let us emphasize that both the Boltzmann factor and the volume form are scalar objects, that does not depend on the choice of coordinates. In this picture the BRST transformation (I.1) is a non-Hamiltonian (and non-local) vector field on the manifold $\mathcal{P}\Gamma$, *i.e.* it is not a symplectic or equivalently a canonical transformation. So the symplectic structure is affected by the BRST vector field (I.1) and it in general becomes non-local.

We see that the BRST-type transformation (I.1) forces us to consider a wider class of theories than the ultra-local ones. To include non-local theories, we have to lift the construction from phase space Γ to a *path* phase space $\mathcal{P}\Gamma$, thereby giving room for non-trivial Poisson bracket between different times. In general, non-local theories are senseless, but we shall see that we can give meaning to a restricted class of non-local theories.

We shall here discard global obstructions and assume that objects like the gauge fermion ψ and the symplectic potential ϑ exist globally. So in particular, our considerations does not include Gribov problems. Also we will for simplicity consider the case where Γ does not have boundaries. Moreover, we assume that the path integral exists for the studied gauges.

Let us introduce a gauge-fixed symplectic two-form directly in the path space $\mathcal{P}\Gamma = \prod_{t\in[0,T]}\Gamma$:

$$\begin{aligned} \omega_{\psi} &= \omega_{\psi}(\gamma) \;=\; \tfrac{1}{2}\int_{\gamma}\int_{\gamma} dt\; dt'\; dz^{A}(t)\;\omega_{\psi AB}(t,t') \;\wedge\; dz^{B}(t') \\ &= \tfrac{1}{2}\int_{\gamma}\int_{\gamma} dt\; dt'\;\omega_{\psi AB}(t,t')\; dz^{B}(t') \;\wedge\; dz^{A}(t)(-1)^{\epsilon_{A}+1}\;. \end{aligned} \tag{II.1}$$

The basic picture is that acting with a BRST type transformation (I.1) change the symplectic two-form depending on the gauge-fixing ψ. In other words, ω_{ψ} is a modified symplectic two-form, whose change is an accumulated effect of acting with a BRST-type of transformations (I.1). We shall derive the explicit formula later. Here we are only interested in the principle idea. The inverse ${\omega_{\psi}}^{AB}(t,t')$ gives rise to a gauge-fixed path space Poisson bracket

$$\{F,G\}_{\psi} \;=\; \{F,G\}_{\psi}(\gamma) \;=\; \int_{\gamma}\int_{\gamma} dt\; dt'\; F\frac{\overleftarrow{\delta^{r}}}{\delta z^{A}(t)}{\omega_{\psi}}^{AB}(t,t')\frac{\overrightarrow{\delta^{l}}}{\delta z^{B}(t')}G\;, \tag{II.2}$$

where $F,G:\mathcal{P}\Gamma\to\mathbb{C}$. For generic ψ the elements $\omega_{\psi AB}(t,t')$ are functionals of the path γ. More precisely, we allow for the following dependence: A) It could be a *functional* of the

full path γ, B) it could be a function of the two path values $\gamma(t)$ and $\gamma(t')$ and finally C) it could depend explicitly on the times t and t'. In symbols,

$$\omega_{\psi AB}(t,t') \;=\; \omega_{\psi AB}(\gamma,\gamma(t),\gamma(t'),t,t') \; . \tag{II.3}$$

As an obvious boundary condition, when the gauge fermion is turned off, $\psi=0$, the gauge-fixed symplectic structure should coincide with the original ultra-local symplectic structure. That is,

$$\omega \;=\; \omega(\gamma) \;=\; \tfrac{1}{2}\int_{\gamma}\int_{\gamma} dt\; dt'\; dz^{A}(t)\; \omega_{AB}(z(t),t)\; \delta(t-t') \;\wedge\; dz^{B}(t') \; . \tag{II.4}$$

In the same manner, instead of looking at classical observables $f:\Gamma \to \mathbb{C}$, let us consider classical observables $F:\mathcal{P}\Gamma \to \mathbb{C}$ that are functions on the *path* space $\mathcal{P}\Gamma$. The usual ultra-local classical observables F are of the form

$$F(\gamma) \;=\; \int_{\gamma} dt\; f(\gamma(t),t) \qquad \text{or} \qquad F(\gamma) \;=\; f(\gamma(t_1),t_1) \tag{II.5}$$

for some function $f:\Gamma\times[0,T] \to \mathbb{C}$ and $t_1\in[0,T]$. Examples are the BRST-improved Hamiltonian $H_{BRST}(z(t),t)$ and the BRST charge $\Omega(z(t),t)$. The path space versions read

$$\begin{aligned} H_{BRST} \;&=\; H_{BRST}(\gamma) \;=\; \int_{\gamma} dt\; H_{BRST}(\gamma(t),t) \; , \\ \Omega \;&=\; \Omega(\gamma) \;=\; \int_{\gamma} dt\; \Omega(\gamma(t),t) \; , \\ \{\Omega,\Omega\} \;&=\; 0 \; , \qquad\qquad \{\Omega, H_{BRST}\} \;=\; 0 \; . \end{aligned} \tag{II.6}$$

The classical observables of the form (II.5) are formally the fundamental building blocks for all classical observables, *i.e.* a classical observable F in path space is a formal powerserie of the type (II.5). As a particular example we allow that the gauge fermion $\psi:\mathcal{P}\Gamma \to \mathbb{C}$ is a powerserie of these building blocks.

III The Space of Gauge Choices

Let us analyze the set of gauge fermions in more detail. First of all, it turns out that we can define a Grassmann-odd bracket structure among the gauge fermions. We have chosen here to start with this construction to emphasize its fundamental nature. We shall later see how this anti-algebra can be realized in the algebra of vector fields as BRST-like vector fields. When exponentiating to the corresponding groups we realize the corresponding BRST flow or – what turn out to be the same – the gauge-fixing flow, see Section IV.

III.1 An Anti-Lie-Bracket

An anti-Lie-bracket in the space $C^{\infty}(\mathcal{P}\Gamma)$ of "classical" observables is defined via the Poisson bracket and the BRST charge:

$$\begin{aligned}(F,G) &= \tfrac{i}{\hbar}(-1)^{\epsilon_F}\{\Omega, FG\} \;=\; \tfrac{i}{\hbar}\{FG,\Omega\}(-1)^{\epsilon_G+1} \\ &= -(-1)^{(\epsilon_F+1)(\epsilon_G+1)}(G,F)\ .\end{aligned} \tag{III.1}$$

The anti-Lie-bracket is of odd Grassmann parity and it has ghost number $+1$. It satisfies the correct symmetry property, the Jacobi identity, but not the Poisson property.

III.2 Group of Gauge Fermions

The space of gauge fermions $\psi : \mathcal{P}\Gamma \to \mathbb{C}$ with ghost number -1 equipped with this anti-Lie-bracket is an infinite dimensional Lie-algebra, that we denote $\mathcal{G}$. The corresponding group G is identified with precisely the same space of ghost number -1 functions $\Psi : \mathcal{P}\Gamma \to \mathbb{C}$, so $G = \mathcal{G}$. The group G is endowed with an associative product $\diamond : G \times G \to G$,

$$\Psi_1 \diamond \Psi_2 \;=\; \Psi_1 + \Psi_2 + \tfrac{i}{\hbar}\Psi_1\{\Omega, \Psi_2\}\ , \tag{III.2}$$

and an inversion map $\tau : G \to G$,

$$\tau(\Psi) \;=\; -\frac{\Psi}{1+\frac{i}{\hbar}\{\Omega,\Psi\}} \;=\; -\frac{\Psi}{1+B} \;=\; -\,\Psi\, e^{-b}\ . \tag{III.3}$$

We have for convenience introduced

$$b \equiv \tfrac{i}{\hbar}\{\Omega,\psi\} = \ln(1+B)\ , \qquad B \equiv \tfrac{i}{\hbar}\{\Omega,\Psi\} = e^{b}-1\ . \tag{III.4}$$

The algebra elements are in general denoted by a lowercase ψ and the group element are denoted by uppercase Ψ. They are connected via the *bijective* exponential map $\mathrm{Exp} : \mathcal{G} \to G$:

$$\Psi = \mathrm{Exp}(\psi) = \psi\ e(b)\ , \qquad \psi = \mathrm{Ln}(\Psi) = \Psi\ l(B)\ , \tag{III.5}$$

where e and l are

$$e(b) \;=\; \int_0^1 d\alpha\ e^{\alpha b} \;=\; \frac{e^b-1}{b} \;=\; \frac{B}{b} \;=\; \sum_{n=1}^{\infty}\frac{b^{n-1}}{n!}\ , \tag{III.6}$$

$$l(B) \;=\; \frac{\ln(1+B)}{B} \;=\; \frac{b}{B} \;=\; \sum_{n=1}^{\infty}\frac{(-B)^{n-1}}{n}\ . \tag{III.7}$$

The neutral element of the Lie group is $\Psi = 0$:

$$\Psi \diamond 0 = \Psi = 0 \diamond \Psi\ , \qquad \mathrm{Exp}(0) = 0\ . \tag{III.8}$$

The exponential map obeys the Baker-Campbell-Hausdorff formula:

$$\begin{aligned} \mathrm{BCH}(u\psi_1, u\psi_2) &= \mathrm{Ln}(\mathrm{Exp}(u\psi_1) \diamond \mathrm{Exp}(u\psi_2)) \\ &= u\psi_1 + u\psi_1 + \tfrac{1}{2}(u\psi_1, u\psi_2) + \mathcal{O}(u^3) \ , \\ \mathrm{Exp}(-\psi) &= \tau(\mathrm{Exp}(\psi)) \ . \end{aligned} \tag{III.9}$$

This is easiest to prove in an algebra/group representation – a so-called realization – that we describe in the next Section.

III.3 Group of Diffeomorphisms

Consider the infinite dimensional Lie group $\mathrm{Diff}(\mathcal{P}\Gamma)$ of diffeomorphisms $\sigma : \mathcal{P}\Gamma \to \mathcal{P}\Gamma$. The corresponding infinite dimensional Lie algebra $\mathrm{Lie}(\mathrm{Diff}(\mathcal{P}\Gamma))$ is identified with the set of bosonic vector fields $X : C^\infty(\mathcal{P}\Gamma) \to C^\infty(\mathcal{P}\Gamma)$ with the usual Lie bracket

$$[X, Y][F] \ = \ X[Y[F]] - Y[X[F]] \ . \tag{III.10}$$

Here we have adapted the common definition [5] of a vector field X that it is a linear derivations on the space of functions $C^\infty(\mathcal{P}\Gamma)$, *i.e.* that it satisfies a graded Leibnitz' rule. Although this definition has the correct transformation properties under change of coordinate patches and it works in the general case, it has a complication that a more elementary definitions of a vector field often doesn't have: It relates naturally to the *reversed* group $(\mathrm{Diff}(\mathcal{P}\Gamma), \overset{\mathrm{op}}{\circ})$ of opposite ordering, where $\sigma_1 \overset{\mathrm{op}}{\circ} \sigma_2 = \sigma_2 \circ \sigma_1$. (Or vice-versa, the Lie-algebra of vector fields acting from the *right* corresponds to the group of usual ordering.) To see this, let us write down the pertinent exponential map Exp from the algebra $(\mathrm{Lie}(\mathrm{Diff}(\mathcal{P}\Gamma)), [\cdot, \cdot])$ to the group $(\mathrm{Diff}(\mathcal{P}\Gamma), \overset{\mathrm{op}}{\circ})$. It is defined as follows. For each vector field $X \in \mathrm{Lie}(\mathrm{Diff}(\mathcal{P}\Gamma))$, there exists a unique one-parameter family solution $u \mapsto \sigma_X(u)$, to the first-order differential equation

$$\left\{ \begin{aligned} \tfrac{d}{du}\sigma_X(u) &= X_{\sigma_X(u)} \\ \sigma_X(u=0) &= \mathrm{id}_{\mathcal{P}\Gamma} \end{aligned} \right. \tag{III.11}$$

In greater detail, this means that for all path γ and functions $f \in C^\infty(\mathcal{P}\Gamma)$

$$\frac{d}{du} f \circ \sigma_X(u, \gamma) \ = \ X_{\sigma_X(u,\gamma)}[f] \ . \tag{III.12}$$

We shall not try to justify in this infinite dimensional problem that for a fixed path γ such solutions $\sigma_X(u, \gamma)$ exists in some interval $u \in [-\epsilon(\gamma), \epsilon(\gamma)]$. And even worse, these solutions should exists uniformly in γ over path space $\mathcal{P}\Gamma$ for $u \in [0, 1]$. To do this properly, we should give a rigorous definition of an infinite dimensional manifold $\mathcal{P}\Gamma$ of paths. But this would take us to far away from the main scope of this paper. We shall simply assume this can be done. Then the exponential map Exp is defined as $\mathrm{Exp}(X) \equiv \sigma_X(u{=}1)$. One may show that

$$\sigma_X(u) \circ \sigma_X(v) \ = \ \sigma_X(u+v) \ , \qquad \sigma_X(u) \ = \ \mathrm{Exp}(uX) \ ,$$

$$\mathrm{Exp}(uX)\circ\mathrm{Exp}(vX) \;=\; \mathrm{Exp}((u+v)X)\ , \qquad \mathrm{Exp}(X)^{-1} \;=\; \mathrm{Exp}(-X)\ . \quad \text{(III.13)}$$

Note that (III.12) implies that

$$\frac{d}{du}(\mathrm{Exp}(uX)^* f)|_{u=0} \;=\; X[f] \;\equiv\; \mathcal{L}_X f\ . \qquad \text{(III.14)}$$

Together with the boundary condition this yields formally

$$f\circ\mathrm{Exp}(uX) \;\equiv\; \mathrm{Exp}(uX)^* f \;=\; \sum_{n=0}^{\infty}\frac{1}{n!}(u\mathcal{L}(X))^n f \;\equiv\; e^{u\mathcal{L}(X)} f\ . \qquad \text{(III.15)}$$

Hence, the formulas for the pull-back and the push-forward read

$$\begin{aligned}(\mathrm{Exp}(X))^* &= e^{\mathcal{L}(X)}\ ,\\ (\mathrm{Exp}(X))_* Y &= Y\circ\sum_{n=0}^{\infty}\frac{1}{n!}X^n \;\equiv\; Y\circ e^{\circ X}\ .\end{aligned} \qquad \text{(III.16)}$$

Moreover, from the reversing property of the pull-back $\sigma_1^*\sigma_2^* = (\sigma_2\circ\sigma_1)^*$, it follows that

$$\mathrm{Exp}(Y)\circ\mathrm{Exp}(X) \;=\; \mathrm{Exp}(X)\overset{\mathrm{op}}{\circ}\mathrm{Exp}(Y) \;=\; \mathrm{Exp}(\mathrm{BCH}(X,Y))\ . \qquad \text{(III.17)}$$

This shows that the Lie algebra of vector fields which acts to the left is naturally associated with the Lie group of *opposite* ordering.

The exponential map $\mathrm{Exp}: \mathrm{Lie}(\mathrm{Diff}(\mathcal{P}\Gamma)) \to \mathrm{Diff}(\mathcal{P}\Gamma)$ gives rise to a right action conventionally denoted by a dot

$$.\ :\ \mathcal{P}\Gamma\times\mathrm{Lie}(\mathrm{Diff}(\mathcal{P}\Gamma)) \to \mathcal{P}\Gamma\ . \qquad \text{(III.18)}$$

Its definition and main properties are

$$\gamma.X = (\mathrm{Exp}(X))\gamma\ , \qquad \gamma.0 = \gamma\ , \qquad (\gamma.X).Y = \gamma.(\mathrm{BCH}(X,Y))\ . \qquad \text{(III.19)}$$

In particular, if the path space $\mathcal{P}\Gamma$ is a right $\mathrm{Lie}(\mathrm{Diff}(\mathcal{P}\Gamma))$ module, *i.e.* a $C^\infty(\mathcal{P}\Gamma)$ vector space endowed with an action

$$[\cdot,\cdot] : \mathcal{P}\Gamma\times\mathrm{Lie}(\mathrm{Diff}(\mathcal{P}\Gamma)) \to \mathcal{P}\Gamma \qquad \text{(III.20)}$$

satisfying

$$\begin{aligned}[][\gamma, F\ X + G\ Y] &= F(\gamma)[\gamma,X] + G(\gamma)[\gamma,Y]\ ,\\ [a\gamma_1 + b\gamma_2, X] &= a[\gamma_1,X] + b[\gamma_2,X]\ ,\\ [[\gamma,X],Y] - [[\gamma,Y],X] - [\gamma,[X,Y]] &= 0\ ,\end{aligned} \qquad \text{(III.21)}$$

then the exponential map can be understood by simple Taylor expansion:

$$(\mathrm{Exp}(X))\gamma \;=\; \gamma e^{[\ \cdot\ ,X]} \;=\; \sum_{n=0}^{\infty}\frac{1}{n!}\underbrace{[[\ldots,[\gamma,X]\ldots,X],X]}_{n\ \text{commutators}}\ . \qquad \text{(III.22)}$$

However, it is unnecessary to assume this.

IV The Gauge-Fixing Flow

We can now construct a Lie algebra homomorphism $X : \mathcal{G} \to \mathrm{Lie}(\mathrm{Diff}(\mathcal{P}\Gamma))$ that takes a gauge fermion ψ to a bosonic BRST-type vector field X^{ψ} by

$$X : \psi \mapsto X^{\psi} \equiv \tfrac{i}{\hbar}\psi X_{\Omega} \ , \qquad [X^{\psi_1}, X^{\psi_2}] = X^{(\psi_1,\psi_2)} \ . \tag{IV.1}$$

The composition of two such vector fields is remarkably again a vector field of the same type

$$X^{\psi_1} \circ X^{\psi_2} \ = \ X^{\psi_1\{\psi_2,\Omega\}} \ , \tag{IV.2}$$

because the derivatives of second order cancels. The corresponding Lie group homomorphism

$$\sigma \ \equiv \ \mathrm{Exp} \circ X \circ \mathrm{Ln} \ : \ G \to \mathrm{Diff}(\mathcal{P}\Gamma) \tag{IV.3}$$

reads

$$\begin{aligned} \sigma : \ \Psi \ \ \mapsto \ \ \sigma_{\Psi} \ &= \ \mathrm{id}_{\mathcal{P}\Gamma} + \tfrac{i}{\hbar}\Psi X_{\Omega} \ , \\ \sigma_{\Psi_1 \diamond \Psi_2} \ &= \ \sigma_{\Psi_1} \overset{\mathrm{op}}{\circ} \sigma_{\Psi_2} \ = \ \sigma_{\Psi_2} \circ \sigma_{\Psi_1} \ , \\ \sigma_{\tau(\Psi)} \ &= \ (\sigma_{\Psi})^{-1} \ . \end{aligned} \tag{IV.4}$$

This is called a realization of G. As a very important consequence we have a gauge-fixing right group action

$$\diamond \ : \ \mathcal{P}\Gamma \times G \to \mathcal{P}\Gamma \tag{IV.5}$$

directly on the path space:

$$\begin{aligned} \gamma_{\Psi} \ = \ \gamma \diamond \Psi \ &= \ \sigma_{\Psi}(\gamma) \ = \ (\mathrm{id}_{\mathcal{P}\Gamma} + \tfrac{i}{\hbar}\Psi X_{\Omega})\,\gamma \ , \\ \gamma \diamond 0 \ &= \ \gamma \ , \\ (\gamma \diamond \Psi_1) \diamond \Psi_2 \ &= \ \gamma \diamond (\Psi_1 \diamond \Psi_2) \ . \end{aligned} \tag{IV.6}$$

This is the sought-for geometric gauge-fixing. The gauge fixing can be understood at the level of paths as a modification of the individual paths!

Again, the expression $\sigma_{\Psi} = \mathrm{id}_{\mathcal{P}\Gamma} + \frac{i}{\hbar}\Psi X_{\Omega}$ is quite symbolic, but it is justified by the formula for the pull-back:

$$(\sigma_{\Psi})^{*} \ = \ (\mathrm{id}_{\mathcal{P}\Gamma} + \tfrac{i}{\hbar}\Psi X_{\Omega})^{*} \ = \ 1 + \tfrac{i}{\hbar}\mathcal{L}(\Psi X_{\Omega}) \ . \tag{IV.7}$$

The expression can be evaluated directly term by term on the manifold if the path space $\mathcal{P}\Gamma$ is a module wrt. the Lie algebra $(C^{\infty}(\mathcal{P}\Gamma), \{\cdot,\cdot\})$.

Before we carry on, let us for the sake of completeness also define the map

$$\begin{aligned} \sigma : \ \psi \ \ \mapsto \ \ \sigma_{\psi} \ &= \ \mathrm{Exp}(X^{\psi}) \ = \ \mathrm{Exp}(\tfrac{i}{\hbar}\psi X_{\Omega}) \ : \ \mathcal{G} \ \to \ \mathrm{Diff}(\mathcal{P}\Gamma) \ , \\ (\sigma_{\psi})^{-1} \ &= \ \sigma_{-\psi} \ , \end{aligned} \tag{IV.8}$$

and the corresponding right "gauge-fixing action" $\mathcal{P}\Gamma \times \mathcal{G} \to \mathcal{P}\Gamma$ on the path space $\mathcal{P}\Gamma$ from the Lie algebra $\mathcal{G}$:

$$\begin{aligned} \gamma.\psi \ &= \ \sigma_{\psi}(\gamma) \ = \ (\mathrm{Exp}(\tfrac{i}{\hbar}\psi X_{\Omega}))\gamma \ , \\ \gamma.0 \ &= \ \gamma \ , \\ (\gamma.\psi_1).\psi_2 \ &= \ \gamma.(\mathrm{BCH}(\psi_1,\psi_2)) \ . \end{aligned} \tag{IV.9}$$

IV.1 Gauge-Fixed Quantities

The gauge-fixed classical observables F_ψ are defined via the pull-back of the gauge-fixing map:

$$\begin{aligned} F_\psi(\gamma) &= (\sigma_\psi^* F)(\gamma) \; = \; F(\sigma_\psi(\gamma)) \\ &= F(\gamma) + \tfrac{i}{\hbar}\Psi(\gamma)\ \{\Omega(\gamma), F(\gamma)\} \\ &= e^{\frac{i}{\hbar}\psi(\gamma)\{\Omega(\gamma),\cdot\}} F(\gamma)\ . \end{aligned} \tag{IV.10}$$

The BRST-improved Hamiltonian H_{BRST} and the BRST charge Ω are not changed by the gauge fixing flow, because of (II.6). The gauge-fixed symplectic two-form and the symplectic potential are remarkably simple, but clearly non-local:

$$\omega_\Psi \; = \; \sigma_\Psi^* \omega \; = \; \omega + \tfrac{i}{\hbar} d\Psi \ \wedge \ d\Omega\ , \tag{IV.11}$$

$$\vartheta_\Psi \; = \; \sigma_\Psi^* \vartheta \; = \; \vartheta + \tfrac{i}{\hbar}\Psi \ d\Omega + \tfrac{i}{\hbar} d[\Psi \ i_{X_\Omega}\vartheta]\ . \tag{IV.12}$$

The gauge-fixed Poisson bracket satisfies

$$\{F_\Psi, G_\Psi\}_\Psi \; = \; (\{F,G\})_\Psi\ \ , \tag{IV.13}$$

or written more explicitly

$$\begin{aligned} \{F,G\}_\Psi &= \sigma_\Psi^* \left(\{\sigma_{\tau(\Psi)}^* F, \sigma_{\tau(\Psi)}^* G\}\right) \\ &= \{F,G\} - \tfrac{i}{\hbar}\{F,\Psi\}\frac{1}{1+B}\{\Omega,G\} - \tfrac{i}{\hbar}\{F,\Omega\}\frac{1}{1+B}\{\Psi,G\} \\ &\quad + (\tfrac{i}{\hbar})^2\{F,\Omega\}\frac{\{\Psi,\Psi\}}{(1+B)^2}\{\Omega,G\} \\ &= \{F,G\} - \tfrac{i}{\hbar}\{F,\psi e(b)\}e^{-b}\{\Omega,G\} - \tfrac{i}{\hbar}\{F,\Omega\}e^{-b}\{e(b)\psi,G\} \\ &\quad + (\tfrac{i}{\hbar})^2\{F,\Omega\}e^{-b}\{e(b)\psi,\psi e(b)\}e^{-b}\{\Omega,G\}\ . \end{aligned} \tag{IV.14}$$

These formulas follows by straightforward application of (IV.10). The derivation becomes somewhat easier if one first derive (IV.14) for $G=F$ being Grassmann-odd. Then one may use the polarization formula $\{F,G\} = \frac{1}{2}(\{F+G,F+G\} - \{F,F\} - \{G,G\})$ and finally linearity to extend to functions of both Grassmann-parities. An even simpler way is the following matrix-calculation. Define, cf. (IV.11) :

$$\begin{aligned} \omega_{\Psi AB}(t,t') &= \omega_{AB}(t,t') + \Delta\omega_{AB}(t,t')\ , \\ \Delta\omega_{AB}(t,t') &= \tfrac{i}{\hbar}\Big(\frac{\overrightarrow{\delta^l}}{\delta z^A(t)}\Psi\Big)\Big(\Omega\frac{\overleftarrow{\delta^r}}{\delta z^B(t')}\Big) + \tfrac{i}{\hbar}\Big(\frac{\overrightarrow{\delta^l}}{\delta z^A(t)}\Omega\Big)\Big(\Psi\frac{\overleftarrow{\delta^r}}{\delta z^B(t')}\Big)\ . \end{aligned} \tag{IV.15}$$

Then the inverse matrix is

$$\begin{aligned} [\omega_\Psi]^{-1} &= ([1] + [\omega]^{-1}[\Delta\omega])^{-1}[\omega]^{-1} \\ &= \sum_{n=0}^{\infty}\left(-[\omega]^{-1}[\Delta\omega])^{-1}\right)^n [\omega]^{-1}\ . \end{aligned} \tag{IV.16}$$

It is easy to see that this sums up to (IV.14) with $F = z^A(t)$ and $G = z^B(t')$ substituted.

The BRST-closed (and therefore BRST-exact) quantities are constant under the BRST gauge-fixing flow. A BRST exact quantity

$$\{\Omega, F\} \;=\; \{\Omega, F_\psi\}_\psi \;, \qquad \Omega_\psi \;=\; \Omega \;, \tag{IV.17}$$

can be written BRST-exactly wrt. the gauge-fixed structure. As an important consequence the BRST cohomology is not affected by the gauge-fixing.

V The BFV Path Integral

In this Section we shall formally prove the BFV Theorem in the path space formulation, by inspecting the various parts of the path integral. We have already argued that the BRST improved Hamiltonian H_{BRST} is not modified by the gauge fixing flow. Below we shall analyze the measure part and the kinetic part.

V.1 The Volume Form

Let us denote the volume form by $\Omega_{\rm vol} = \mathcal{D}z \; {\rm Pf}(\omega_{\cdot\cdot})$. The gauge-fixed volume form can remarkably be written as

$$\sigma_\psi^* \Omega_{\rm vol} \;=\; \Omega_{\rm vol} \; \frac{1}{1+B} \;=\; \Omega_{\rm vol} \; e^{-\frac{i}{\hbar}\{\Omega,\psi\}} \;. \tag{V.1}$$

It is perhaps surprising that the gauge-fixing term, which can be view as a way to remove zero-directions in the Hessian of the action, from a geometric point of view has little to do with the action. In fact, the above shows that it emerges from the measure part, completely dictated by the gauge-fixing flow. Because of the importance of (V.1), let us indicate a proof in the case of a finite number $2n$ of purely bosonic variables $z^A(t) = z^I$, where $(A,t) = I = 1,\ldots,2n$, so that

$$\Omega_{\rm vol} = \frac{1}{n!} \; \omega^{\wedge \; n} \;=\; \frac{1}{n!} \; \omega \wedge \ldots \wedge \omega \;. \tag{V.2}$$

If there are fermionic variables present, the volume form is no longer the n'th exterior power of the symplectic two-form, and the analysis becomes more involved. However, let's stick to the bosonic case. The gauge-fixed top-form is, cf. (IV.11),

$$\begin{aligned}
\sigma_\Psi^* \Omega_{\rm vol} \;&=\; \frac{1}{n!} \; \sigma_\Psi^* \omega \wedge \ldots \wedge \sigma_\Psi^* \omega \\
&=\; \sum_{r=0}^{n} \frac{1}{(n-r)! r!} \; \omega^{\wedge \; (n-r)} \wedge (\tfrac{i}{\hbar} d\Psi \;\wedge\; d\Omega)^{\wedge \; r} \\
&=\; \sum_{r=0}^{n} \frac{1}{(n-r)! r!} \; \omega^{\wedge \; (n-r)} \;\wedge \sum_{I_1,\ldots,I_r} \; (\tfrac{i}{\hbar} dz^{I_1} \frac{\delta \Psi}{\delta z^{I_1}} \;\wedge\; d\Omega) \;\wedge
\end{aligned}$$

$$\ldots \wedge \ (\tfrac{i}{\hbar} dz^{I_r} \frac{\delta \Psi}{\delta z^{I_r}} \ \wedge \ d\Omega) \ . \tag{V.3}$$

Because of covariance, one may resort to Darboux coordinates $z^I = (q^i, p_i), \ i = 1, \ldots, n$. Then

$$\{q^i, p_j\} \ = \ \delta^i_j \ , \qquad\qquad \omega \ = \ dp_i \wedge dq^i \ . \tag{V.4}$$

A simple combinatorial argument yields that

$$\begin{aligned} \sigma^*_\Psi \Omega_{\rm vol} \ &= \ (dp_1 \ \wedge \ dq^1) \ \wedge \ldots \wedge \ (dp_n \ \wedge \ dq^n) \\ &\qquad \times \sum_{r=0}^{n} \sum_{i_1, \ldots, i_r} (-\tfrac{i}{\hbar})(\frac{\delta \Psi}{\delta q^{i_1}} \frac{\delta \Omega}{\delta p_{i_1}} - \frac{\delta \Psi}{\delta p_{i_1}} \frac{\delta \Omega}{\delta q^{i_1}}) \times \\ &\qquad \ldots \times (-\tfrac{i}{\hbar})(\frac{\delta \Psi}{\delta q^{i_r}} \frac{\delta \Omega}{\delta p_{i_r}} - \frac{\delta \Psi}{\delta p_{i_r}} \frac{\delta \Omega}{\delta q^{i_r}}) \\ &= \ \Omega_{\rm vol} \sum_{r=0}^{n} (-B)^r \ = \ \Omega_{\rm vol} \frac{1 - (-B)^{n+1}}{1+B} \ . \end{aligned} \tag{V.5}$$

Now we formally take the $n = \infty$ limit to derive (V.1).

V.2 The Kinetic Part

We have already analyzed the variation of the Hamiltonian part and the measure under the gauge-fixing flow. Let us now draw our attention to the last piece of the path integral, namely the kinetic term. It is handy to introduce a vector field T that calculates the difference between the total time derivative and the explicit time derivative:

$$T \ \equiv \ \frac{d}{dt} - \frac{\partial}{\partial t} \ = \ \int_\gamma dt \ \dot{\gamma}^A(t) \frac{\overrightarrow{\delta^l}}{\delta z^A(t)} \quad : \quad C^\infty(\mathcal{P}\Gamma) \to C^\infty(\mathcal{P}\Gamma) \ . \tag{V.6}$$

The kinetic term can be written as the contraction of the symplectic potential with this vector field:

$$i_T \vartheta(\gamma) = \int_\gamma dt \ \vartheta_A(t) \dot{\gamma}^A(t) \ , \qquad \vartheta(\gamma) = \int_\gamma dt \ \vartheta_A(t) dz^A(t) \ . \tag{V.7}$$

We can now write the gauge-fixed BFV path integral as

$$\mathcal{Z}_\psi \ = \ \int \Omega_{\rm vol} \ e^{\frac{i}{\hbar}(i_T \vartheta - H_{BRST} - \{\Omega, \psi\})} \ . \tag{V.8}$$

We would like to only sum over paths in the path integral so that the kinetic term for these paths is left invariant under the gauge-fixing flow. In fact, if this is fulfilled, we have realized gauge-fixing as an internal diffeomorphism of the path integral.[1] Thus the path integral cannot depend on the gauge fermion and we can conclude the BFV Theorem. To achieve

[1] Perhaps it is helpful a this point to recall that every integral $I = \int f(x) \ dx = \int f(\sigma(x)) \ d\sigma(x)$ by the very definition is diffeomorphism invariant. x is just a dummy variable.

this conclusion, one should restrict the path space by only allowing for paths satisfying the following BRST boundary condition

$$(i_{X_\Omega}\vartheta)(\gamma)|_{t=0} + \Omega(\gamma)|_{t=0} \;=\; (i_{X_\Omega}\vartheta)(\gamma)|_{t=T} + \Omega(\gamma)|_{t=T} \;. \qquad \text{(V.9)}$$

Because of the formula

$$i_{X_\Omega}\vartheta(\gamma) \;=\; \int_\gamma dt\; \vartheta_A(t) X^A_\Omega(t) \qquad \text{(V.10)}$$

and because Ω is BRST-invariant, we have

$$\begin{aligned} X_\Omega[i_{X_\Omega}\vartheta](\gamma) \;&=\; \int_\gamma\int_\gamma dt\; dt'\; X^B_\Omega(t') \frac{\overrightarrow{\delta^l}}{\delta z^B(t')}[\vartheta_A(t)\; X^A_\Omega(t)] \\ &=\; \tfrac{1}{2}\int_\gamma dt\; [X_\Omega, X_\Omega]^A(t)\; \vartheta_A(t)(-1)^{\epsilon_A} \\ &\quad + \int_\gamma\int_\gamma dt\; dt'\; X^A_\Omega(t)\; X^B_\Omega(t')\; \frac{\overrightarrow{\delta^l}}{\delta z^B(t')}\vartheta_A(t) \\ &=\; \tfrac{1}{2}\int_\gamma dt\; X^A_{\{\Omega,\Omega\}}(t)\; \vartheta_A(t)(-1)^{\epsilon_A} \\ &\quad + \tfrac{1}{2}\int_\gamma\int_\gamma dt\; dt'\; X^A_\Omega(t)\; X^B_\Omega(t')\; \omega_{BA}(t', t) \;=\; 0 \;. \end{aligned} \qquad \text{(V.11)}$$

Together with $X_\Omega[\Omega] = 0$ this yields that the above boundary condition on the paths is stabile under the gauge-fixing action of $\mathcal{G}$. We will assume that a vector field

$$X \;:\; \gamma \mapsto X_\gamma \;=\; \int_\gamma dt\; X^A(t) \frac{\overrightarrow{\delta^l}}{\delta z^A(t)} \qquad \text{(V.12)}$$

has the following dependence on the path γ:

$$X^A(t) \;=\; X^A(\gamma, \gamma(t), t) \;. \qquad \text{(V.13)}$$

Under the assumption that the original symplectic potential $\vartheta_A(t) = \vartheta_A(\gamma(t))$ is ultra-local and that it has no explicit time dependence (and therefore the symplectic two-form is also of that kind), one derives that a generic vector field X changes the kinetic term with

$$X[i_T\vartheta](\gamma) \;=\; i_X\vartheta|_{\gamma,\gamma(t=T),t=T} - i_X\vartheta|_{\gamma,\gamma(t=0),t=0} + \int_\gamma dt\; X^A(t)\; \omega_{AB}(t)\; \dot{z}^B(t) \;. \qquad \text{(V.14)}$$

Assuming that Ω is ultra-local and that it has no explicit time dependence and using the boundary condition (V.9), we therefore conclude that the kinetic term is invariant under the gauge-fixing flow $X = \psi X_\Omega$.

V.3 Infinitesimal Gauge-Fixing Flow

Above we have argued from a formal point of view that turning on or turning off gauge-fixing does not change the value of the path integral. In real applications, like quantum field

theory, turning off gauge-fixing makes the path integral ill-defined, as a result of integrating over an infinite gauge-volume. That is why gauge-fixing is introduced in the first place. It is therefore desirable to have an infinitesimal analysis.

Consider the map

$$L_{\mathrm{Exp}} \;:\; \mathcal{G} \to \mathrm{Aut}(G) \;:\; \psi \mapsto L_{\mathrm{Exp}(\psi)} \ , \tag{V.15}$$

where L_Ψ is the left multiplication map with a fixed Ψ

$$L_\Psi(\Psi') \;=\; \Psi \diamond \Psi' \ . \tag{V.16}$$

Then by definition

$$L_{\mathrm{Exp}(0)} \;=\; \mathrm{id}_G \tag{V.17}$$

Let us consider an infinitesimal change $\psi \to \psi + \delta\psi$ in the gauge fermion. The relevant object is the differential of L_{Exp} in $\psi = 0$,

$$(L_{\mathrm{Exp}})_{*0} \;:\; T_0\mathcal{G} \to T_{\mathrm{id}_G}\mathrm{Aut}(G) \ . \tag{V.18}$$

It reads

$$\Psi + \delta\Psi \;=\; ((L_{\mathrm{Exp}})_{*0}\ \delta\psi)\,(\Psi) \;=\; h_\Psi(\delta\psi) \diamond \Psi \ . \tag{V.19}$$

Here the function $h : \mathcal{G} \times T_0\mathcal{G} \to \mathcal{G}$ is

$$\begin{aligned} h_\psi(\delta\psi) &= (\Psi + \delta\Psi) \diamond \tau(\Psi) \;=\; \frac{\delta\Psi}{1+B} \;=\; \frac{\mathrm{Exp}_{*\psi}(\delta\psi)}{1+B} \\ &= e^{-b}e(b)\ \delta\psi + e^{-b}e'(b)\ \psi\ \{\Omega, \delta\psi\} \\ &= \frac{1-e^{-b}}{b}\delta\psi + \frac{b-1+e^{-b}}{b^2}\psi\ \{\Omega, \delta\psi\} \ . \end{aligned} \tag{V.20}$$

We have identified the tangent space $T_{\mathrm{id}_G}\mathrm{Aut}(G)$ with maps from the group G that to each element Ψ of the group G give an element in $\mathcal{G} \diamond \Psi$.

So the infinitesimal change in the gauge fixing flow $\sigma_{\psi+\delta\psi} \circ \sigma_{-\psi}$ is governed by a BRST-type vector field, cf. (IV.6),

$$X^{h_\psi(\delta\psi)} \;=\; h_\psi(\delta\psi) X_\Omega \ . \tag{V.21}$$

Note that

$$\{\Omega, h_\psi(\mu)\} = \{\Omega, \mu\} \ , \qquad h_{\alpha\psi}(\psi) = \alpha\ \psi \ . \tag{V.22}$$

We can now derive the infinitesimal change of various objects under an infinitesimal change $\psi \to \psi + \delta\psi$ of the gauge fermion caused by the vector field $X^{h_\psi(\delta\psi)}$. For instance, the measure changes according to

$$\mathrm{div}_\rho X^{h_\psi(\delta\psi)} \;=\; -\tfrac{i}{\hbar}\{\Omega, \delta\psi\} \ . \tag{V.23}$$

This is precisely the infinitesimal change in the gauge-fixed action.

VI The Sp(2) Case

The whole construction can easily be extended to the $Sp(2)$ case[4]. We shall thus be brief and just state the construction.

In the $Sp(2)$ case we have two odd BRST charges Ω^a, $a = 1, 2$, that are mutual orthogonal in the Poisson bracket sense:

$$\{\Omega^a, \Omega^b\} \ = \ 0 \ . \tag{VI.1}$$

We define a Grassmann-even bracket

$$[F, G] \ = \ F \ k(G) - k(F) \ G \ = \ -(-1)^{\epsilon_F \epsilon_G}[G, F] \ , \tag{VI.2}$$

where k is a Grassmann-even nilpotent second order operator

$$\begin{aligned} k \ &= \ \tfrac{i}{2\hbar}\epsilon_{ab} X_{\Omega^a} \circ X_{\Omega^b} \ , \qquad k^2 \ = \ 0 \ , \\ k(F) \ &= \ \tfrac{i}{2\hbar}\epsilon_{ab}\{\Omega^a, \{\Omega^b, F\}\} \ = \ \tfrac{i}{2\hbar}\{\{F, \Omega^a\}, \Omega^b\}\epsilon_{ab} \ . \end{aligned} \tag{VI.3}$$

The Grassmann-even bracket satisfies the usual symmetry property, the Jacobi identity, but not the Poisson property. We define a degenerate metric structure (a covariant 2-tensor)

$$\begin{aligned} g(F, G) \ &= \ k(F) \ G + F \ k(G) - k(FG) \ = \ \tfrac{i}{\hbar}\{F, \Omega^a\}\epsilon_{ab}\{\Omega^b, G\} \\ &= \ +(-1)^{\epsilon_F \epsilon_G} g(G, F) \ . \end{aligned} \tag{VI.4}$$

k and g satisfy the following relations

$$g(k(F), G) \ = \ 0 \ , \qquad k(g(F, G)) \ = \ 2 \ k(F) \ k(G) \ , \tag{VI.5}$$

$$g(F, g(G, H)) \ = \ \tfrac{1}{2} g(F, G) \ k(H) + \tfrac{1}{2} g(F, H) \ k(G)(-1)^{\epsilon_G \epsilon_H} \ . \tag{VI.6}$$

We have a Jacobi-like identity:

$$g(F, g(G, H)) - (-1)^{\epsilon_F \epsilon_G} g(G, g(F, H)) \ = \ g([F, G], H) \ . \tag{VI.7}$$

The gauge is now chosen by specifying a gauge *boson*[2] ϕ. The even bracket $[\cdot, \cdot]$ is a Lie-bracket in the algebra $\bar{\mathcal{G}}$ of gauge bosons. The space $\bar{G} = \bar{\mathcal{G}}$ of gauge bosons becomes a Lie group by introducing a product and an inversion

$$\begin{aligned} \Phi_1 \diamond \Phi_2 \ &= \ \Phi_1 + \Phi_2 + \Phi_1 \ k(\Phi_2) \ , \\ \tau(\Phi) \ &= \ -\frac{\Phi}{1 + \bar{B}} \ = \ -\ \Phi \ e^{-\bar{b}} \ , \end{aligned} \tag{VI.8}$$

where

$$\begin{aligned} \bar{b} \ &= \ \tfrac{i}{2\hbar}\epsilon_{ab}\{\Omega^a, \{\Omega^b, \phi\}\} \ = \ k(\phi) \ = \ \ln(1 + \bar{B}) \ , \\ \bar{B} \ &= \ \tfrac{i}{2\hbar}\epsilon_{ab}\{\Omega^a, \{\Omega^b, \Phi\}\} \ = \ k(\Phi) \ = \ e^{\bar{b}} - 1 \ , \end{aligned} \tag{VI.9}$$

[2] Our gauge boson is *minus a half* of the BLT gauge boson.

and ϕ and Φ are connected via the bijective exponential map $\mathrm{Exp}:\bar{\mathcal{G}}\to\bar{G}$

$$\Phi = \mathrm{Exp}(\phi) = \phi\ e(\bar{b})\ , \qquad \phi = \mathrm{Ln}(\Phi) = \Phi\ l(\bar{B})\ . \tag{VI.10}$$

We can define a homomorphism $\Psi_a:\bar{\mathcal{G}}\to\mathcal{G}_a$ that transforms a gauge boson ϕ into a gauge fermion[3] $\Psi_a(\phi)$ of the kind $a=1,2$ (corresponding to the generator Ω^a)

$$\begin{aligned}
\Psi_a(\cdot) &= \epsilon_{ab}X_{\Omega^b}(\cdot) \\
\Psi_a(F) &= \epsilon_{ab}\{\Omega^b,F\} \ = \ (-1)^{\epsilon_F}\{F,\Omega^b\}\epsilon_{ba}\ , \\
\Psi_a\left([\phi_1,\phi_2]\right) &= \left(\Psi_a(\phi_1),\Psi_a(\phi_2)\right)^a\ , \\
\Psi_a\left(\Phi_1\diamond\Phi_2\right) &= \Psi_a(\Phi_1)\diamond\Psi_a(\Phi_2)
\end{aligned} \tag{VI.11}$$

(no sum over a.). Note that

$$\tfrac{i}{\hbar}\{\Omega^a,\Psi_b(F)\} \ = \ \delta^a_b\ k(F)\ . \tag{VI.12}$$

The algebra homomorphism $\bar{X}:\bar{\mathcal{G}}\to\mathrm{Lie}(\mathrm{Diff}(\mathcal{P}\Gamma))$ is the sum of the two BRST homomorphisms

$$\bar{X}:\phi\mapsto\bar{X}^{\phi}\equiv g(\phi,\cdot)=\tfrac{i}{\hbar}\Psi_a(\phi)X_{\Omega^a}\ , \qquad [\bar{X}^{\phi_1},\bar{X}^{\phi_2}]=\bar{X}^{[\phi_1,\phi_2]}\ . \tag{VI.13}$$

The corresponding Lie group homomorphism $\bar{\sigma}\equiv\mathrm{Exp}\circ\bar{X}\circ\mathrm{Ln}:\bar{G}\to\mathrm{Diff}(\mathcal{P}\Gamma)$ reads

$$\begin{aligned}
\bar{\sigma}:\ \Phi\ \ \mapsto\ \ \bar{\sigma}_{\Phi} &= \mathrm{id}_{\mathcal{P}\Gamma}+g(\Phi,\cdot)+\tfrac{1}{2}g(\Phi,\Phi)\ k(\cdot)\ , \\
\bar{\sigma}_{\Phi_1\diamond\Phi_2} &= \bar{\sigma}_{\Phi_1}\overset{\mathrm{op}}{\circ}\bar{\sigma}_{\Phi_2}\ , \\
\bar{\sigma}_{\tau(\Phi)} &= (\bar{\sigma}_{\Phi})^{-1}\ .
\end{aligned} \tag{VI.14}$$

The pull-back map $(\bar{\sigma}_{\Phi})^*$ is

$$(\bar{\sigma}_{\Phi})^* \ = \ 1+\tfrac{i}{\hbar}\Psi_a(\Phi)\ \mathcal{L}(X_{\Omega^a})+\tfrac{i}{2\hbar}\epsilon_{ab}\ g(\Phi,\Phi)\ \mathcal{L}(X_{\Omega^a})\circ\mathcal{L}(X_{\Omega^b})\ . \tag{VI.15}$$

The gauge-fixed symplectic two-form reads

$$\omega_{\Phi} \ = \ \bar{\sigma}^*_{\Phi}\omega \ = \ \omega+\tfrac{i}{\hbar}d\Psi_a(\Phi)\ \wedge\ d\Omega^a\ . \tag{VI.16}$$

The gauge-fixed volume form reads

$$\bar{\sigma}^*_{\phi}\Omega_{\mathrm{vol}} \ = \ \Omega_{\mathrm{vol}}\ \frac{1}{(1+\bar{B})^2} \ = \ \Omega_{\mathrm{vol}}\ e^{-\frac{i}{\hbar}\epsilon_{ab}\{\Omega^a,\{\Omega^b,\phi\}\}}\ . \tag{VI.17}$$

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[3] To be precise, $\Psi_a(\phi)=\frac{1}{2}\psi$ is *half* the corresponding gauge fermion of the standard BFV-BRST construction.

References

[1] C. Becchi, A. Rouet and R. Stora, Comm. Math. Phys. **42** (1975) 127; Ann. Phys. (N.Y.) **98** (1976) 287; I.V. Tyutin, *Gauge Invariance in Field Theory and in Statistical Physics in the Operator Formalism*, Lebedev preprint FIAN No. 39 (1975), unpublished.

[2] E.S. Fradkin and G.A. Vilkovisky, Phys. Lett. **B55** (1975) 224.
I.A. Batalin and G.A. Vilkovisky, Phys. Lett. **B69** (1977) 309.
E.S. Fradkin and T.E. Fradkina, Phys. Lett. **B72** (1978) 343.
I.A. Batalin and E.S. Fradkin, Phys. Lett. **B122** (1983) 157.

[3] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems*, Princeton University Press, 1992; M. Henneaux, Phys. Rep. **126** (1985) 1.

[4] I.A. Batalin, P.M. Lavrov and I.V. Tyutin, J. Math. Phys. **31** (1990) 6; *ibid.* **31** (1990) 2708; Int. J. Mod. Phys. **A6** (1991) 3599.

[5] M. Nakahara, *Geometry, Topology and Physics*, (IOP Publishing Ltd, Bristol) 1990.